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THE ANALYTIC CONTINUATION OF THE DISCRETE SERIES. I

BY

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ABSTRACT. In this paper the analytic continuation of the holomorphic discrete series is defined. The most elementary properties of these representations are developed. The study of when these representations are unitary is begun.

1. Introduction. In Sally [8] (among other things), the analytic continuation of the holomorphic discrete series for the universal covering group of $SL(2, \mathbf{R})$ was studied. In this paper we generalize the results of Sally [8] to an arbitrary simply connected, semisimple, Lie group admitting (relative) holomorphic discrete series. We also show that the characters of the analytically continued representations are holomorphic functions of the parameter. This allows the use of the formula of Harish-Chandra [4] for the characters of discrete series to compute, in particular, the characters of the "limits of discrete series" in Knapp-Okamoto [6]. In Knapp-Okamoto [6] it is shown that these "limits of discrete series" are irreducible components of unitarily induced representations. Thus the results of this paper give a technique for computing the characters of irreducible components of certain unitarily induced representations. In particular for $SU(n, 1)$ it gives a technique for computing the characters of the irreducible components of an infinite class of reducible unitary principal series and, for $SU(1, 1)$ and $SU(2, 1)$, all of them.

In the course of our investigation we note that, in addition to the "limits of discrete series", there are unitary representations "past the limit" just as in the case of the universal covering group of $SL(2, \mathbf{R})$ (see Lemma 3.5 and §4).

Most of this paper is of an expository nature. We use some ideas of Murakami and Satake [9] to give the "bounded realization" of the holomorphic discrete series. In the proof of Proposition 2.6 and Lemma 2.7 this realization is shown to be equivalent to that of Harish-Chandra [3]. The critical observation in §2 is Lemma 2.5. §§3 and 4 contain whatever might be new in this paper.

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In a later paper in this series we will study the analytic continuation of the "nonholomorphic" discrete series for the universal covering group of $SU(n, 1)$.

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2. The holomorphic discrete series. Let G be a connected, simply connected, simple Lie group. Let $\mathfrak{g}G = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra, $\mathfrak{g}$, of G. We assume that $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}] \neq \mathfrak{k}$. Then $\mathfrak{k} = \mathbf{R}iH_1 \oplus \mathfrak{k}_1$, $[H_1, \mathfrak{k}_1] = 0$ (here $\mathfrak{g} \subset \mathfrak{g}_c$ the complexification of $\mathfrak{g}$). Let $\mathfrak{h}_* \subset \mathfrak{k}$ be a maximal abelian subalgebra of $\mathfrak{k}$ and let $\mathfrak{h}$ denote its complexification. Let Δ be the root system of $\mathfrak{g}_c$ relative to $\mathfrak{h}$. Then $\mathfrak{k}_c = \mathfrak{k} \otimes_R \mathbf{C} = \mathfrak{h} + \Sigma_{\alpha \in \Delta_K} \mathfrak{g}_\alpha$, $\mathfrak{p}_c = \Sigma_{\alpha \in \Delta_P} \mathfrak{g}_\alpha$ (we use these formulas to define Δ_K and Δ_p). Then, as is well known, if $\alpha \in \Delta_P$, $\alpha(H_1) = \pm c$, $c \in R$, $c \neq 0$. We normalize H_1 so that $\alpha(H_1) = \pm 1$ for $\alpha \in \Delta_P$. Let Δ^+ be a positive system of roots relative to a lexicographic order starting with H_1. Since $\alpha(H_1) = 0$ for $\alpha \in \Delta_K$, we see that if $\alpha \in \Delta_K$ and $\beta \in \Delta_P^+ = \Delta_P \cap \Delta^+$, then $\alpha + \beta \in \Delta$ implies $\alpha + \beta \in \Delta_P^+$.

Let G_c be the connected, simply connected group with Lie algebra $\mathfrak{g}_c$. Then G_c is a complex, simple, Lie group. Let $\mathfrak{n}^+ = \Sigma_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \Sigma_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$. Set $\mathfrak{h}_R = \{H \in \mathfrak{h} | \alpha(H) \in \mathbf{R},\ \alpha \in \Delta\}$. Set $A^+ = \exp \mathfrak{h}_R$, $N^+ = \exp \mathfrak{n}^+$, $N^- = \exp \mathfrak{n}^-$. We recall the following results of Harish-Chandra (cf. Helgason [5]).

(1) Set $P^+ = \exp \mathfrak{p}^+$, $P^- = \exp \mathfrak{p}^-$ where $\mathfrak{p}^+ = \mathfrak{p}_c \cap \mathfrak{n}^+$, $\mathfrak{p}^- = \mathfrak{p}_c \cap \mathfrak{n}^-$. Then $P^-K_cP^+$ is open in G_c, and the map $P^- \times K_c \times P^+ \to G_c$ is a holomorphic diffeomorphism of $P^- \times K_c \times P^+$ onto an open subset of G_c. Here K_c is the connected subgroup of G_c corresponding to $\mathfrak{k}_c$.

(2) If $G_0 \subset G_c$ is the connected subgroup of G_c corresponding to $\mathfrak{g} \subset \mathfrak{g}_c$, then $N^-A^+G_0$ is an open subset of $P^-K_cP^+$. $P^-K_c \cap G_0 = K_0$ is the connected subgroup of G_0 corresponding to $\mathfrak{k}$.

If $g \in P^-K_cP^+$, let $\tilde{\mathfrak{K}}(g)$ be defined by $g \in P\tilde{\mathfrak{K}}(g)P^+$. Then $\tilde{\mathfrak{K}}$: $P^-K_cP^+ \to K_c$ is a holomorphic map.

(3) Let $\gamma_1 \in \Delta_P^+$ be the smallest element. Let $\Phi_1 \subset \Delta_P^+$ be the set of all $\alpha \in \Delta_P^+$, $\alpha \neq \gamma_1$ such that $\gamma_1 \pm \alpha \notin \Delta$. If $\Phi_1 \neq \varnothing$ let γ_2 be the smallest element of Φ_1. Set $\Phi_2 = \{\alpha \in \Phi_1 - \{\gamma_2\} | \gamma_2 \pm \alpha \notin \Delta\}$. Continuing in this way we have $\Delta_P^+ \supset \Phi_1 \supset \Phi_2 \supset \cdots \supset \Phi_r$ and $\Phi_{r+1} = \varnothing$. We also have $\gamma_1, \ldots, \gamma_r$ such that $\gamma_i \pm \gamma_j \notin \Delta$. If $\alpha \in \Delta_P$, let $X_\alpha \in \mathfrak{g}_\alpha$ be chosen so that $\overline{X}_\alpha = X_{-\alpha}$. (Here $\overline{X} = X_1 - iX_2$ for $X_1, X_2 \in \mathfrak{g}$.) Then, if $\mathfrak{a} = \Sigma_{i=1}^r R(X_{\gamma_i} + X_{-\gamma_i})$, $\mathfrak{a} \subset \mathfrak{p}$ is maximal abelian.

(4) Normalize X_α for $\alpha \in \Delta_P$ so that $[X_\alpha, X_{-\alpha}] = H_\alpha$, $[H_\alpha, X_\alpha] = 2X_\alpha$ and $[H_\alpha, X_{-\alpha}] = -2X_{-\alpha}$. Then

$$\exp\Big(\sum t_i(X_{\gamma_i} + X_{-\gamma_i})\Big) = \exp\Big(\sum (\tanh t_i) X_{-\gamma_i}\Big) \cdot \exp\Big(\sum (\log \cosh t_i) H_{\gamma_i}\Big) \exp\Big(\sum (\tanh t_i) X_{\gamma_i}\Big).$$

(5) Using the fact that $G_0 = K_0(\exp \mathfrak{a})K_0$, we see that $N^-A^+G_0 = P^-K_c\exp(\Omega)$ with $\Omega = \{\mathrm{Ad}(k)(\sum \tanh t_iX_{\gamma_i}) | k \in K, t_i \in R\}$.

Let $\nu: G \to G_0$ be the covering map. Then G acts on Ω by $P^-K_c\exp(z \cdot g) = P^-K_c(\exp z)\nu(g)$, $z \in \Omega$, $g \in G$. This is the Harish-Chandra realization of G/K as a bounded homogeneous domain in C^n, $n = \dim \mathfrak{p}^+ = \frac{1}{2} \dim G/K$. Here, K is the connected subgroup of G corresponding to $\mathfrak{k}$.

Let $\tilde{K}_c \xrightarrow{\gamma} K_c$ be the universal covering group of K_c. Following Satake [9], we define $\tilde{\mathcal{K}}: \Omega \times G \to K_c$ by the formula $\tilde{\mathcal{K}}(z: g) = \tilde{\mathcal{K}}((\exp z)\nu(g))$ for $z \in \Omega$, $g \in G$. $\tilde{\mathcal{K}}$ lifts to a holomorphic map $\mathcal{K}$ of $\Omega \times G$ to $\tilde{K}_c$.

Since G_c is simply connected, there is a Lie group isomorphism, $g \to \bar{g}$ such that $\overline{(\exp X)} = \exp \bar{X}$. We define for $z_1, z_2 \in \Omega$,

$$\tilde{D}(z_1 : z_2) = \tilde{\mathcal{K}}\left((\exp z_2)(\exp \bar{z}_1)^{-1}\right)^{-1}.$$

Then $\tilde{D}: \Omega \times \Omega \to K_c$. Hence $\tilde{D}$ lifts to a map $D: \Omega \times \Omega \to \tilde{K}_c$.

(6) $D(z_1 \cdot g: z_2 \cdot g) = \overline{\mathcal{K}(z_1 : g)}^{-1} D(z_1 : z_2)\mathcal{K}(z_2 : g)$.

LEMMA 2.1.

$$D\left(\mathrm{Ad}(k)\left(\sum_{i=1}^{r} t_iX_{\gamma_i}\right): \mathrm{Ad}(k)\left(\sum_{i=1}^{r} t_iX_{\gamma_i}\right)\right) = \exp\left(-\mathrm{Ad}(k)^{-1}\left(\sum_{i=1}^{r} \log(1 - t_i^2)H_{\gamma_i}\right)\right)$$

for $-1 < t_i < 1, i = 1, \ldots, r, k \in K$.

PROOF.

$$D\left(\mathrm{Ad}(k)\left(\sum_{i=1}^{r} t_iX_{\gamma_i}\right): \mathrm{Ad}(k)\left(\sum_{i=1}^{r} t_iX_{\gamma_i}\right)\right) = k^{-1}D\left(\sum_{i=1}^{r} t_iX_{\gamma_i}: \sum_{i=1}^{r} t_iX_{\gamma_i}\right)k$$

by (6).

$$\exp\left(\sum t_iX_{\gamma_i}\right)\overline{\exp\left(\sum t_iX_{\gamma_i}\right)}^{-1} = \exp\left(\sum t_iX_{\gamma_i}\right)\exp\left(-\sum t_iX_{-\gamma_i}\right).$$

Now $[X_{\gamma_i}, X_{\gamma_j}] = [X_{\gamma_i}, X_{-\gamma_j}] = 0$ if $i \neq j$. Note that $CX_{\gamma_i} + CH_{\gamma_i} + CX_{-\gamma_i}$ is isomorphic to the Lie algebra of $SL(2, \mathbf{C})$ with

$$X_{\gamma_i} \leftrightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_{-\gamma_i} \leftrightarrow \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad H_{\gamma_i} \leftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Under this identification, we see that

$$\exp(tX_{\gamma_i})\exp(-tX_{-\gamma_i}) \leftrightarrow \begin{bmatrix} 1 - t^2 & t \\ -t & 1 \end{bmatrix}$$

$$\leftrightarrow \exp\left(-\frac{t}{1 - t^2} X_{-\gamma_i}\right)\exp\left(\log(1 - t^2)H_{\gamma_i}\right)\exp\left(\frac{t}{1 - t^2} X_{\gamma_i}\right).$$

The lemma follows from these observations.

LEMMA 2.2. *If $f \in C_0(\Omega)$ (continuous with compact support) then, for any $g \in G$,*

$$\int_\Omega f(z \cdot g)\, dz = \int_\Omega f(z) |\det(\operatorname{Ad} \mathcal{K}(z: g^{-1})|_{\mathfrak{p}^+})|^{-2}\, dz$$

where dz is Lebesgue measure on $\mathfrak{p}^+$.

PROOF. We note that $\exp: \mathfrak{p}^+ \to P^+$ is a holomorphic diffeomorphism and since $[\mathfrak{p}^+, \mathfrak{p}^+] = 0$ we see that, if $f \in C_0(P^+)$, then

$$\int_{\mathfrak{p}^+} f(\exp z)\, dz = \int_{P^+} f(p^+)\, dp^+,$$

where dp^+ is a Haar measure on P^+.

Thus, if $f \in C_0(\Omega)$, we may identify f with an element of $C_0(P^+)$. We note that if all measures are properly normalized, and if $F \in C_0(G_c)$ then

$$\int_{P^-K_cP^+} F(g)\, dg = \int_{P^- \times K_c \times P^+} F(p^- k p^+) |\det(\operatorname{Ad}(k)|_{\mathfrak{p}^+})|^2\, dp^+\, dk\, dp^+.$$

Let $F \in C_0(G_c)$ be such that

$$\int_{P^- \times K_c} F(p^- k \exp z)\, dp^-\, dk = f(z) \quad \text{for all } z \in \Omega.$$

Then

$$\begin{aligned}
\int_\Omega f(z \cdot g) dz &= \int_{P^- \times K_c \times \Omega} F(p^- k \exp(z) \nu(g))\, dp^- dk\, dz \\
&= \int_{P^- K_c \exp \Omega} F(x \nu(g)) |\det((\operatorname{Ad} \mathcal{K}(x))|_{\mathfrak{p}^+})|^{-2}\, dx \\
&= \int_{P^- K_c \exp \Omega} F(x) |\det(\operatorname{Ad}(\mathcal{K}(x \nu(g)^{-1}))|_{\mathfrak{p}^+})|^{-2}\, dx \\
&= \int_{P^- \times K_c \times \Omega} F(p^- k \exp z) |\det(\operatorname{Ad}(k)|_{\mathfrak{p}^+})|^2 \\
&\quad \cdot |\det(\operatorname{Ad}(\mathcal{K}(p^- k(\exp z)) \nu(g)^{-1}))|^{-2}\, dp^-\, dk\, dz \\
&= \int_\Omega f(z) |\det(\operatorname{Ad}(\mathcal{K}(z: g^{-1}))|_{\mathfrak{p}^+})|^{-2}\, dz. \quad \text{Q.E.D.}
\end{aligned}$$

LEMMA 2.3. *The G-invariant measure on Ω is*

$$d\mu(z) = \det(\operatorname{Ad}(D(z : z)|_{\mathfrak{p}^+}))\, dz.$$

PROOF. Let $f \in C_0(\Omega)$. Then

$$\begin{aligned}
\int_\Omega f(z \cdot g) d\mu(z) &= \int_\Omega f(z \cdot g) \det(\operatorname{Ad}(D(z : z))|_{\mathfrak{p}^+})\, dz \\
&= \int_\Omega f(z) |\det(\operatorname{Ad} \mathcal{K}(z : g^{-1})|_{\mathfrak{p}^+})|^{-2} \det(\operatorname{Ad}(D(z \cdot g^{-1}: z \cdot g^{-1}))|_{\mathfrak{p}^+})\, dz.
\end{aligned}$$

Now $D(z \cdot g^{-1} : z \cdot g^{-1}) = \overline{\mathcal{K}(z : g^{-1})}^{-1} D(z : z)\mathcal{K}(z : g^{-1})$. If $k \in K_c$, then $k = \exp itH_1 \cdot k_1$ with $k_1 \in [K_c, K_c]$ and $\bar{k} = \exp i\bar{t}H_1 \cdot \bar{k}_1$. Now $\det(\mathrm{Ad}(K_1)|_{\mathfrak{p}^+}) = \det(\mathrm{Ad}(\bar{k}_1)|_{\mathfrak{p}^+}) = 1$ and $\det(\mathrm{Ad}(k)|_{\mathfrak{p}^+}) = e^{int}$ ($n = \dim_C \mathfrak{p}^+$). Hence

$$|\det \mathrm{Ad}(k)|_{\mathfrak{p}^+}|^{-2} \det\big(\mathrm{Ad}(\bar{k})|_{\mathfrak{p}^+}\big)^{-1} \det(\mathrm{Ad}(k)|_{\mathfrak{p}^+}) = e^{2n \operatorname{Im} t} e^{-2n \operatorname{Im} t} = 1.$$

This proves the lemma.

LEMMA 2.4. *If $(\pi, v, \langle\ ,\ \rangle)$ is a finite dimensional unitary representation of K extended to $\tilde{K}_c$ as a holomorphic representation then $\pi(D(z : z))$ is a positive definite operator for $z \in \Omega$.*

PROOF. If $X \in \mathfrak{k}_c$ then $\langle \pi(x)v, w\rangle = -\langle v, \pi(\bar{x})w\rangle$, $v, w \in V$. Thus, if $H \in i\mathfrak{h}_*$, then $\langle \pi(H)v, w\rangle = \langle v, \pi(H)w\rangle$. The result now follows from Lemma 2.1.

Let $\pi = \{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots for Δ^+. We may assume that $\Delta_K^+ \cap \pi = \{\alpha_2, \dots, \alpha_l\}$. Let $\Lambda_0 \in \mathfrak{h}^*$ be such that

(*) $2\langle \Lambda_0, \alpha_j\rangle / \langle \alpha_j, \alpha_j\rangle$ is a nonnegative integer for $j = 2, \dots, l$ and equal to 0 if $j = 1$.

Let (π_0, V^{Λ_0}) be the irreducible unitary representation of K with highest weight Λ_0. Let $\langle \Lambda_1, \alpha_j\rangle = 0, j = 2, \dots, l$, $2\langle \Lambda_1, \alpha_1\rangle / \langle \alpha_1, \alpha_1\rangle = 1$. Then Λ_1 is the differential of a character of K, which we denote e^{Λ_1}. Furthermore $e^{\lambda\Lambda_1}$ is defined for all $\lambda \in C$, since K is simply connected. Thus $e^{z\Lambda_1} \otimes \pi_0$ defines an irreducible holomorphic representation of $\tilde{K}_c$ for all $z \in C$.

Let $\mathcal{H}^{\pi_0}$ be the space of all $f : \Omega \to V^{\pi_0}$ such that f is holomorphic. If $g \in G$ and $f \in \mathcal{H}^{\pi_0}$, define for $\lambda \in C$

$$\big(T_{\pi_0,\lambda}(g)f\big)(z) = \big(e^{\lambda\Lambda_1} \otimes \pi_0\big)(\mathcal{K}(x : g))f(x \cdot g).$$

Then it is easily seen that

$$T_{\pi_0,\lambda}(g_1 g_2) = T_{\pi_0,\lambda}(g_1) T_{\pi_0,\lambda}(g_2).$$

Furthermore we have

LEMMA 2.5. *$T_{\pi_0,\lambda}|_K = (e^{\lambda\Lambda_1} \otimes T_{\pi_0,0})|_K$. Also $T_{\pi_0,0}(g) = T_{\pi_0,0}(g')$ if $\nu(g) = \nu(g')$. That is $T_{\pi_0,0}|_K$ is actually a representation of K_0.*

If $f \in \mathcal{H}^{\pi_0}$ define, for $\lambda \in R$,

$$\|f\|^2_{\pi_0,\lambda} = \int_\Omega \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z : z)) f(z), f(z)\rangle \, d\mu(z).$$

Lemma 2.4 implies that the right-hand side of the above formula is positive if it converges. Let $H^{\pi_0,\lambda}$ be the space of all $f \in H^{\pi_0}$ such that $\|f\|^2_{\pi_0,\lambda} < \infty$. Let $\langle\ ,\ \rangle_{\pi_0,\lambda}$ denote the associated inner product on $H^{\pi_0,\lambda}$.

PROPOSITION 2.6. *If* $f \in H^{\pi_0,\lambda}$ *then* $\|T_{\pi_0,\lambda}(g)f\|_{\pi_0,\lambda} = \|f\|_{\pi_0,\lambda}$ *for all* $g \in G$. *If* $H^{\pi_0,\lambda} \neq (0)$ *then the constant functions are in* $H^{\pi_0,\lambda}$. *Furthermore* $H^{\pi_0,\lambda}$ *is complete and if* $H^{\pi_0,\lambda} \neq (0)$ *then the polynomial functions* $f\colon \Omega \to V^{\pi_0}$ *are dense in* $H^{\pi_0,z}$. *Finally* $(T_{\pi_0,\lambda}, H^{\pi_0,\lambda})$ *is a unitary representation of* G.

PROOF.

$$\begin{aligned}
\|T_{\pi_0,\lambda}(g)f\|^2_{\pi_0,\lambda} &= \int_\Omega \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z:z)) \\
&\qquad \cdot (e^{\lambda\Lambda_1} \otimes \pi_0)(\mathcal{K}(z:g)f(z\cdot g)), f(z\cdot g)\rangle\, d\mu(z), \\
&= \int_\Omega \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(\overline{\mathcal{K}(z:g)})^{-1}(e^{\lambda\Lambda_1} \otimes \pi_0)(D(z:z)) \\
&\qquad \cdot (e^{\lambda\Lambda_1} \otimes \pi_0)(\mathcal{K}(z:g))f(z\cdot g), f(z\cdot g)\rangle\, d\mu(z) \\
&= \int_\Omega \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z\cdot g : z\cdot g))f(z\cdot g), f(z\cdot g)\rangle\, d\mu(z) \\
&= \int_\Omega \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z:z))f(z), f(z)\rangle\, d\mu(z) = \|f\|^2_{\pi_0,\lambda}.
\end{aligned}$$

This proves the first assertion.

If $f \in \mathcal{H}^{\pi_0}$ and $\gamma \in \hat{K}_0$ let

$$\begin{aligned}
f_\gamma(z) &= d(\gamma)\int_{K_0} \overline{\chi_\gamma(k)}\, \pi_0(k)f(z\cdot k)\, dk \\
&= d(\gamma)\int_{K_0} \overline{\chi_\gamma(k)}\, \pi_0(k)f(\mathrm{Ad}(k)^{-1}z)\, dk.
\end{aligned}$$

Here χ_γ is the character of γ and $d(\gamma)$ is the dimension. The integral defining f_γ clearly converges uniformly on compact subsets of Ω. Thus $f_\gamma \in \mathcal{H}^{\pi_0}$. Furthermore $f = \sum f_\gamma$ with uniform and absolute convergence on compact subsets of Ω. Using the Stone-Weierstrass theorem on an arbitrary $\mathrm{Ad}(K_0)$ invariant open subset ω of Ω so that $\bar{\omega} \subset \Omega$, we see that f_γ is a polynomial mapping of Ω to V^{Λ_0}.

Let us now choose, for each $j \in Z$, $j > 0$, $\Omega_j \subset \Omega$ so that $\bar{\Omega}_j$ is compact, $\mathrm{Ad}(K_0)\Omega_j \subset \Omega_j$, $\Omega_{j+1} \supset \Omega_j$, and $\bigcup_{j=1}^{\infty}\Omega_j = \Omega$. (For example take $\Omega_j = \{\mathrm{Ad}(k_0)(\sum_{i=1}^{r} t_i X_{\gamma_i}) | k_0 \in K, |t_i| < j/(j+1)\}$.) Then

$$\begin{aligned}
&\int_{\Omega_j} \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z:z))f(z), f(z)\rangle\, d\mu(z) \\
&\qquad = \int_{\Omega_j} \sum_{\gamma\in\hat{K}} \langle (e^{z\Lambda_1} \otimes \pi_0)(D(z:z))f_\gamma(z), f_\gamma(z)\rangle\, d\mu(z)
\end{aligned}$$

by Schur orthogonality.

The Lebesgue monotone convergence theorem implies that

$$\|f\|^2_{\pi_0,\lambda} = \lim_{j\to\infty} \int_{\Omega_j} \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z:z))f(z), f(z)\rangle d\mu(z).$$

Suppose that $f \in H^{\pi_0,\lambda}$. Then, setting $\pi = e^{\lambda\Lambda_1} \otimes \pi_0$, we have

$$\int_{\Omega_j} \langle \pi(D(z:z))f(z), f(z)\rangle \, d\mu(z) \geq \int_{\Omega_j} \langle \pi(D(z:z))f_\gamma(z), f_\gamma(z)\rangle \, d\mu(z)$$

for all j. Hence $f_\gamma \in H^{\pi_0,\lambda}$. This also implies that if $H^{\pi_0,\lambda} \neq (0)$ then there are polynomials in $H^{\pi_0,\lambda}$ and the set of polynomials in $H^{\pi_0,\lambda}$ is dense.

The fact that $(T_{\pi_0,\lambda}, H^{\pi_0,\lambda})$ is a (continuous) unitary representation is standard once we have proved that $H^{\pi_0,\lambda}$ is complete. To see that $H^{\pi_0,\lambda}$ is complete, we note that if $\omega \subset \Omega$ is any compact subset then there is $C_\omega > 0$ so that

$$\|f\|^2_{\pi_0,z} \geq C_\omega \int_\omega \|f(z)\|^2 \, dz.$$

The completeness follows from

$$\int_\omega \|f(z)\|^2 \, dz \geq C'_\omega \sup_{z\in\omega} \|f(z)\| \quad \text{with } C'_\omega > 0$$

for ω the closure of an open subset of Ω, ω compact (cf. Helgason [4, Chapter 8]). We have therefore shown that $(T_{\pi_0,\lambda}, H^{\pi_0,\lambda})$ is a unitary representation of G.

Using the fact that the polynomial functions in $H^{\pi_0,\lambda}$ are dense, we see that, if $H^{\pi_0,\lambda}_\gamma = \{f_\gamma | f \in H^{\pi_0,\lambda}\}$, then $H^{\pi_0,\lambda}_\gamma$ is finite dimensional. Using results of Harish-Chandra [1], we see that, if $H^{\pi_0,\lambda}_F = \Sigma_{\gamma\in K_0} H^{\pi_0,\lambda}_\gamma$ (algebraic direct sum), then

$$T_{\pi_0,\lambda}(X)f = \frac{d}{dt}\left(T_{\pi_0,\lambda}(\exp tX)f\right)\Big|_{t=0}$$

defines a representation of $\mathfrak{g}$ on $H^{\pi_0,\lambda}_F$.

We need the following lemma.

LEMMA 2.7. *Let for* $\gamma \in K_0$, $\mathcal{H}^{\pi_0}_\gamma = \{f_\gamma | f \in \mathcal{H}^{\pi_0}\}$ *(see the second part of the proof of Proposition* 2.5). *Let for* $X \in \mathfrak{g}$,

$$(T_{\pi_0,\lambda}(X)f)(z) = \frac{d}{dt}\left(T_{\pi_0,\lambda}(\exp tX)f\right)(z)\Big|_{t=0} \quad \textit{for } f \in \mathcal{H}^{\pi_0}_\gamma.$$

Then $T_{\pi_0,\lambda}(X)f \in \Sigma_{\gamma\in K_0}\mathcal{H}^{\pi_0}_\gamma = \mathcal{H}^{\pi_0}_F$ *and* $(T_{\pi_0,\lambda}, \mathcal{H}^{\pi_0}_F)$ *is a representation of* $\mathfrak{g}$. *Furthermore, if* $W \subset \mathcal{H}^{\pi_0}_F$ *is a nonzero invariant subspace of* $\mathcal{H}^{\pi_0}_F$, *then* W *contains the constant functions.*

PROOF. Let $f \in \mathcal{H}_F^{\pi_0}$. Then f extends to $P^-\tilde{K}_c \exp \Omega$ (looked upon as the universal covering space of $P^-K_c \exp \Omega$) by defining $f(p^-k \exp z) = (e^{\lambda\Lambda_1} \otimes \pi_0)(k)f(z)$ for $p^- \in P^-$, $k \in \tilde{K}_c$, $z \in \Omega$. Let μ be a nonzero element in the lowest weight space of $(V^{\pi_0})^*$. Let for $f \in \mathcal{H}^{\pi_0}$, $A(f)(g) = \mu(f(g))$ for $g \in P^-\tilde{K}_c\Omega = N^-A^+G$. If $g \in U(\mathfrak{g}_c)$, the universal enveloping algebra of $\mathfrak{g}_c$, let $\tilde{f}(g) = (g \cdot A(f))(e)$. Then $\tilde{f}(zg) = (\lambda\Lambda_1 + \Lambda_0)(z)\tilde{f}(g)$ for $z \in \mathfrak{n}^- \oplus \mathfrak{h}$ (here $(\lambda\Lambda_1 + \Lambda_0)(\mathfrak{n}^-) = 0$). Since the map $f \to \tilde{f}$ is injective (f is holomorphic), the Poincaré-Birkhoff-Witt theorem implies that $\mathcal{H}_F^{\pi_0} = \Sigma_{\Lambda \in \mathfrak{h}^*} \mathcal{H}_\Lambda^{\pi_0}$ with $\Lambda = (\lambda\Lambda_1 + \Lambda_0) - \Sigma n_i\alpha_i$ with $n_i \geq 0$, $n_i \in Z$. Here for $\Lambda \in \mathfrak{h}^*$, $\mathcal{H}_\Lambda^{\pi_0} = \{f \in \mathcal{H}_F^{\pi_0} | T_{\pi_0,\lambda}(h)f = \Lambda(h)f$ for $h \in \mathfrak{h}\}$. Furthermore, if $f \in \mathcal{H}^{\pi_0}$ and $T_{\pi_0,\lambda}(\mathfrak{n}^+)f = 0$, then $f \in \mathcal{H}_{\lambda\Lambda_1+\Lambda_0}^{\pi_0}$. We observe that $\mathcal{H}_{\lambda\Lambda_1+\Lambda_0}^{\pi_0}$ consists of the functions $f(z) = v$ with v in the highest weight space of V^{Λ_0}.

Now $W \subset \mathcal{H}_F^{\pi_0,\lambda}$, $W \neq 0$, implies $W = \Sigma(W \cap \mathcal{H}_\Lambda^{\pi_0,\lambda})$. Let Λ' be such that $W \cap \mathcal{H}_\Lambda^{\pi_0,\lambda} \neq (0)$ and $\Lambda' = \lambda\Lambda_1 + \Lambda_0 - \Sigma n_i\alpha_i$ with Σn_i minimal. Then $T_{\pi_0,\lambda}(\mathfrak{n}^+)(W \cap \mathcal{H}_{\Lambda'}^{\pi_0,\lambda}) = 0$. Hence $\Lambda' = \lambda\Lambda_1 + \Lambda_0$. Thus the constants are in W.

We now conclude the proof of Proposition 2.6. $H_F^{\pi_0,\lambda} \subset \mathcal{H}_F^{\pi_0}$ is $T_{\pi_0,\lambda}$ invariant and nonzero if $H^{\pi_0,\lambda} \neq (0)$. Thus, by the above, $H^{\pi_0,\lambda} \neq (0)$ implies $H^{\pi_0,\lambda}$ contains the constants. Thus, if $H^{\pi_0,\lambda} \neq 0$, then

$$\int_\Omega \langle \pi(D(z:z))v, v\rangle \, d\mu(z) < \infty \quad \text{for all } v \in V^{\Lambda_0}. \tag{1}$$

If f is a polynomial function from Ω to V_0, then f is a linear combination of elements of the form $\varphi \cdot v$ with $\varphi: \Omega \to C$ a polynomial and $v \in V^{\pi_0}$. But

$$\int_\Omega \langle \pi(D(z:z))\varphi(z)v, \varphi(z)v\rangle \, d\mu(z) = \int_\Omega |\varphi(z)|^2 \langle \pi(D(z:z))v, v\rangle \, d\mu(z)$$

$$\leq \sup_{z \in \Omega} |\varphi(z)|^2 \cdot \int_\Omega \langle \pi(D(z:z))v, v\rangle \, d\mu(z) < \infty. \quad \text{Q.E.D.}$$

COROLLARY 2.8. *If* $\lambda \in \mathbf{R}$, *then* $H^{\pi_0,\lambda} \neq (0)$ *if and only if* $\int_\Omega \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z:z))v, v\rangle \, d\mu(z) < \infty$ *for all* $v \in V^{\Lambda_0}$.

COROLLARY 2.9. *If* $(T_{\pi_0,\lambda}, \mathcal{H}_F^{\pi_0})$ *admits an invariant inner product then* $(T_{\pi_0,\lambda}, \mathcal{H}_F^{\pi_0})$ *is irreducible.*

PROOF. Let $W \subset \mathcal{H}_F^{\pi_0}$ be invariant, $W \neq (0)$. Then $W^\perp \subset \mathcal{H}_F^{\pi_0}$ is nonzero if and only if W contains the constants (see Lemma 2.7). Hence $W^\perp = 0$ since W contains the constants.

LEMMA 2.10. *Let* $z \in \mathbf{C}$. *If*

$$\int_\Omega (e^{z\Lambda_1} \otimes \pi_0)(D(z:z)) \, d\mu(z)$$

converges, then it is equal to $d(\pi_0, z)I$ *with* $d(\pi_0, z) \in \mathbf{C}$.

PROOF. $\int_\Omega (ez^{\Lambda_1} \otimes \pi_0)(D(z \cdot u : z \cdot u))\, d\mu(z) = \int_\Omega (e^{z\Lambda_1} \otimes \pi_0)(D(z : z))\, d\mu(z)$ for all $u \in K$. The result now follows from the irreducibility of (π_0, V^{Λ_0}) and (6) (preceding Lemma 2.1).

We observe that more is true.

COROLLARY 2.11.

$$d(\pi_0, z) = \frac{1}{d_{\pi_0}} \int_\Omega e^{z\Lambda_1}(D(z : z))\chi_{\pi_0}(D(z : z))\, d\mu(z)$$

with χ_{π_0} *the character of* (π_0, V^{Λ_0}) *and* $d_{\pi_0} = \dim V^{\Lambda_0}$.

3. The analytic continuation of the holomorphic discrete series. We retain the notation of §2. Let $\mathcal{P}(\mathfrak{p}^+)$ be the space of holomorphic polynomial functions from $\mathfrak{p}^+$ to $\mathbf{C}$. On $\mathfrak{p}^+$ put the inner product $\langle X, Y\rangle = -B(X, \tau Y)$ where B is the Killing form of $\mathfrak{g}_c$ and τ is the conjugation of $\mathfrak{g}_c$ relative to $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$. Extend $\langle\ ,\ \rangle$ to an inner product on $\mathcal{P}(\mathfrak{p}^+) = S((\mathfrak{p}^+)^*)$ in the usual fashion ($S(V)$ denotes the symmetric algebra on the vector space V). Let $\langle\ ,\ \rangle$ also denote a K-invariant inner product on V^{Λ_0}. On $\mathcal{P}(\mathfrak{p}^+) \otimes V^{\Lambda_0}$ put the tensor product the inner product which we also denote $\langle\ ,\ \rangle$. In the following, $\mathcal{P}^j(V)$ will denote the polynomial functions on V homogeneous of degree j.

For the rest of this section (π_0, V^{Λ_0}) will be fixed.

LEMMA 3.1. *If* $f \in \mathcal{P}^j(\mathfrak{p}^+) \otimes V^{\Lambda_0}$ *then*

$$\int_\Omega \langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z : z))f(z), f(z)\rangle\, d\mu(z) = \langle A_j(\lambda)f, f\rangle,$$

converges absolutely for $((\operatorname{Re}\lambda)\Lambda_1 + \Lambda_0 + \rho)(H_\beta) < 0$ *for all* $\beta \in \Delta_P^+$. ($\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$.) *Moreover,* $\lambda \to A_j(\lambda)$ *extends to a rational function from* $\mathbf{C}$ *to* $\operatorname{End}(\mathcal{P}^j(\mathfrak{p}^+) \otimes V^{\Lambda_0})$. *Furthermore the singularities of* $A_j(\lambda)$ *are only at the points* $\lambda \in Z/2$ *(half integers). Finally, if* $\lambda \in \mathbf{R}$ *and if* $A_j(\lambda)$ *is defined then* $A_j(\lambda)$ *is Hermitian.*

PROOF. Let $\mathfrak{h}^+ = \sum_{i=1}^r CH_{\gamma_i}$. Let $\mathfrak{h}^- = \{H \in \mathfrak{h} | \gamma_i(H) = 0$ for $i = 1, \ldots, r\}$. Let $\Delta_0 = \{\alpha \in \Delta | \alpha|_{\mathfrak{h}^+} = 0\}$. Then (up to constants of normalization) if $f \in C_0(\Omega)$

$$\int_\Omega f(z)\, dz = \int_{K_0} \int_{1 > t_1 > t_2 > \cdots > t_r > 0} f\Big(\operatorname{Ad}(k)\Big(\sum_{i=1} t_i X_{\gamma_i}\Big)\Big) \prod_{\alpha\in\Delta^+ - \Delta_0} \Big|\alpha\Big(\sum t_i H_{\gamma_i}\Big)\Big|\, dk\, dt_1 \cdots dt_r. \qquad (1)$$

To prove (1) we note that if $A: \mathfrak{p}^+ \to \mathfrak{p}$ is defined by $A(X) = \frac{1}{2}(X + \bar{X})$ $(\overline{(X_1 + iX_2)} = X_1 - iX_2$ for $X_1, X_2 \in \mathfrak{g})$, then A is a real linear isomorphism such that $A \circ \operatorname{Ad}(k) = \operatorname{Ad}(k) \circ A$ for $k \in K_0$. This implies that $\mathfrak{p}^+ = \operatorname{Ad}(K_0)(\sum_{i=1}^r \mathbf{R}X_{\gamma_i})$. In fact, if $\mathfrak{a} = \sum \mathbf{R}(X_{\gamma_i} + X_{-\gamma_i})$, then $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and $\operatorname{Ad}(K_0)\mathfrak{a} = \mathfrak{p}$ (cf. Helgason [5]). On the other hand,

$$\int_{\mathfrak{p}} f(x)\, dx = \int_{\mathfrak{a}^+} \left(\prod_{\lambda \in \Lambda^+} |\lambda(H)|^{m_\lambda} \right) \left(\int_{K_0} f(\operatorname{Ad}(k)H)\, dk \right) dH \qquad (*)$$

where Λ^+ is the set of positive restricted roots of $(\mathfrak{g}, \mathfrak{a})$ relative to some order, m_λ is the dimension of the λ restricted weight space and $\mathfrak{a}^+$ is a positive Weyl chamber $(\lambda(H) > 0$ for $\lambda \in \Lambda^+)$. $(*)$ can be found in say Helgason [5]. Now (1) follows by applying the Cayley transform (cf. Harish-Chandra [3]).

We now compute

$$\prod_{\alpha \in \Delta^+ - \Delta_0} \left| \alpha\left(\sum t_i H_{\gamma_i} \right) \right|, \qquad t_1, \ldots, t_r \in \mathbf{R},\ t_i > 0.$$

To do this, we recall some results of Harish-Chandra [3].

Let for $1 < i < r$, $C_i = \{\alpha \in \Delta_K^+ | \alpha|_{\mathfrak{h}^+} = -\frac{1}{2}\gamma_i\}$, for $1 < i < j < r$, $C_{ij} = \{\alpha \in \Delta_K^+ | \alpha|_{\mathfrak{h}^+} = \frac{1}{2}(\gamma_j - \gamma_i)\}$. Then

$$\Delta_K^+ = \Delta_0^+ \cup \bigcup_{i=1}^r C_i \cup \bigcup_{1<i<j<r} C_{ij}.$$

Let $P_i = \{\alpha \in \Delta_P^+ | \alpha|_{\mathfrak{h}^+} = \frac{1}{2}\gamma_i\}$, $P_{ij} = \{\alpha \in \Delta_P^+ | \alpha|_{\mathfrak{h}^+} = \frac{1}{2}(\gamma_i + \gamma_j)\}$, $1 < i < j < r$. Then

$$\Delta_P^+ = \{\gamma_1, \ldots, \gamma_r\} \cup \bigcup_{i=1}^r P_i \bigcup_{1<i<j<r} P_{ij}.$$

Let r_i (resp. r_{ij}) be the order of C_i (resp. C_{ij}). Then r_i (resp. r_{ij}) is the order of P_i (resp. P_{ij}). This says that

$$\prod_{\alpha \in \Delta^+ - \Delta_0} \left| \alpha\left(\sum t_i H_{\gamma_i} \right) \right| = C \prod_{i=1}^r |t_i^{2r_i+1}| \prod_{1<i<j<r} |(t_i^2 - t_j^2)|^{r_{ij}}.$$

Furthermore, by the above, $\mathfrak{a}^+$ corresponds to $\{\sum t_i(X_{\gamma_i} + X_{-\gamma_i}),\ t_1 > t_2 > \cdots > t_r > 0\}$. This implies that for $t_1 > t_2 > \cdots > t_r > 0$

$$\prod_{\alpha \in \Delta^+ - \Delta_0} \left| \alpha\left(\sum t_i H_{\gamma_i} \right) \right| = C \prod_{i=1}^r t_i^{2r_i+1} \prod_{1<i<j<r} t_i^2 - t_j^2 = P(t_1, \ldots, t_r).$$

Now (from (1), Lemmas 2.2, 2.3 and $\rho_P = \frac{1}{2}(\text{tr ad } H|_{\mathfrak{p}^+})$)

$$(**) \quad \int_\Omega \left\langle (e^{\lambda\Lambda_1} \otimes \pi_0)(D(z:z))f(z), f(z) \right\rangle d\mu(z)$$

$$= \int_{1>t_1>t_2>\cdots>t_r>0} \prod_{i=1}^r t_i^{2r_i+1} \prod_{1<i<j<r} (t_i^2 - t_j^2)$$

$$\cdot \prod_{i=1}^r (1-t_i^2)^{-(\lambda\Lambda_1+2\rho_p)(H_{\gamma_i})}$$

$$\cdot \Bigg(\int_{K_0} \left\langle \pi_0(k)^{-1}\pi_0\left(\exp\left(-\sum (\log|1-t_i^2|)H_{\gamma_i}\right)\right)\pi_0(k) \right.$$

$$\left. \cdot f\left(\operatorname{Ad}(k)\left(\sum t_i X_{\gamma_i}\right)\right), f\left(\operatorname{Ad}(k)\left(\sum t_i X_{\gamma_i}\right)\right) \right\rangle dk_0 \Bigg) dt_1 \cdots dt_r.$$

The integral over K_0 above is a polynomial $q(t_1, \ldots, t_r)$ in $t_1, \ldots, t_r$. Thus we have

$$(**) = \int_{1>t_1>\cdots>t_r>0} \prod_{i=1}^r (1-t_i^2)^{-(\lambda\Lambda_1+2\rho_p)(H_{\gamma_i})}$$

$$\cdot p(t_1, \ldots, t_r) q(t_1, \ldots, t_r)\, dt_1 \cdots dt_r.$$

As observed in Harish-Chandra [3] such an integral is a rational function of λ (it also clearly converges for $\lambda < 0$ and λ large since $\Lambda_1(H_{\gamma_i}) = 1$, $i = 1, \ldots, r$). The assertion on the pole structure follows from the fact that $2\rho_p(H_{\gamma_i})$ is an integer for $i = 1, \ldots, r$.

The absolute convergence statement follows from the case $f \equiv v \in V^{\Lambda_0}$ in Corollary 2.11 and the work of Harish-Chandra [3]. The last assertion is clear (see Lemma 2.4).

LEMMA 3.2 (HARISH-CHANDRA [3]).

$$d(\pi_0, z)^{-1} = \prod_{\alpha \in \Delta_P^+} \left\{ \frac{-(\Lambda_0 + z\Lambda_1 + \rho)(H_\alpha)}{\rho(H_\alpha)} \right\}$$

up to constants of normalization.

Now let $c(\pi_0) = \operatorname{Sup}\{z \in \mathbf{R} | (z\Lambda_1 + \Lambda_0 + \rho)(H_\alpha) < 0 \text{ for } \alpha \in \Delta_P^+\}$.

Lemma 2.4 combined with Lemma 3.1 and Corollary 2.7 imply that $(T_{\pi_0,z}, H^{\pi_0,z})$ is an irreducible representation of G for $z < c(\pi_0)$. This implies that $(T_{\pi_0,z}, \mathcal{H}_F^{\pi_0})$ is irreducible as a representation of $U(\mathfrak{g}_C)$. However, more is true.

LEMMA 3.3 (HARISH-CHANDRA [2]). *$(T_{\pi_0,z}, \mathcal{H}_F^{\pi_0})$ is irreducible for $z < c(\pi_0)$.*

Combining this with results of Harish-Chandra [2], [3], we have the following lemma.

LEMMA 3.4. *If* $f \in \mathcal{H}_F^{\pi_0}$ *then*

$$\lim_{\substack{\lambda \to c(\pi_0) \\ \lambda < c(\pi_0)}} d_{\pi_0,\lambda}^{-1} \|f\|_{\pi_0,z}^2 = \|f\|_{\pi_0}^2$$

exists and induces a positive definite invariant inner product on $\mathcal{H}_F^{\pi_0}$. *This representation extends to G on the Hilbert space completion of* $\mathcal{H}_F^{\pi_0}$.

Note. This gives another realization of the "limits of holomorphic discrete series" in Knapp-Okamoto [5].

We also note that Lemmas 3.1, 3.4 have the following consequence.

LEMMA 3.5. *Let for* $f \in \mathcal{H}_F^{\pi_0}$,

$$|f|_{\pi_0,\lambda}^2 = d(\pi_0, \lambda)^{-1} \|f\|_{\pi_0,\lambda}^2.$$

Then $\lambda \to |f|_{\pi_0,\lambda}^2$ *is a rational function of* λ *and there is a constant* $\tilde{c}(\pi_0) > c(\pi_0)$ *so that* $|\cdots|_{\pi_0,\lambda}^2$ *defines a positive definite invariant inner product on* $\mathcal{H}_F^{\pi_0}$ *for* $\lambda \in R$ *and* $\lambda < \tilde{c}(\pi_0)$.

In §4 we will determine $\tilde{c}(1)$ for $G_0 = SU(n, 1)$.

Let $\mathcal{H}_F^{\pi_0}$ be as in Lemma 2.7. Then $\mathcal{H}_F^{\pi_0} = \Sigma_{\gamma \in \hat{K}_0} \mathcal{H}_\gamma^{\pi_0}$ where $\mathcal{H}_\gamma^{\pi_0} = \{f_\gamma | f \in \mathcal{H}^{\pi_0}\}$ and f_γ is as in the proof of Proposition 2.6. Let E_γ: $\mathcal{H}^{\pi_0} \to \mathcal{H}_\gamma^{\pi_0}$ be defined (as in the proof of Proposition 2.6) by

$$(E_\gamma f)(z) = d(\gamma) \int_{K_0} \overline{\chi_\gamma(k)}\ \pi_0(k) f(\mathrm{Ad}(k)^{-1} z)\, dk. \tag{1}$$

If $g \in G$, $\gamma \in \hat{K}_0$ define

$$\phi_\gamma^{\pi_0,\lambda}(g) = \mathrm{tr}(E_\gamma T_{\pi_0,\lambda}(g)|_{\mathcal{H}_\gamma^{\pi_0}}). \tag{2}$$

LEMMA 3.6. *The function* $\mathbf{C} \times G \to C$ *given by* $(\lambda, g) \mapsto \phi_\gamma^{\pi_0,\lambda}(g)$ *is continuous and holomorphic in* λ.

PROOF.

$$\begin{aligned}(E_\gamma T_{\pi_0,\lambda}(g) f)(z) &= d(\gamma) \int_{K_0} \overline{\chi_\gamma(k)}\, (T_{\pi_0,\lambda}(g) f)(\mathrm{Ad}(k)^{-1} z)\, dk \\ &= d(\gamma) \int_{K_0} \overline{\chi_\gamma(k)}\, (e^{\lambda\Lambda_1} \otimes \pi_0)(\mathcal{K}(z \cdot k : g)) f(z \cdot kg)\, dk \\ &= d(\gamma) \int_{K_0} \overline{\chi_\gamma(k)}\ e^{\lambda\Lambda_1}(\mathcal{K}(zk : g))(\pi_0(\mathcal{K}(z \cdot k : g)) f(z \cdot kg))\, dk.\end{aligned}$$

If $\mu \in (V^{\Lambda_0})^*$ and $z \in \Omega$ are fixed, then the above computation clearly implies that $(\lambda, g) \mapsto \mu((E_\gamma T_{\pi_0,\lambda}(g) f)(z))$ satisfies the continuity and holomorphy properties asserted for $\phi_\gamma^{\pi_0,\lambda}$. It is easily seen that, if $(\mu \otimes \varepsilon_z) f = \mu(f(z))$, then the set $\{\mu \otimes \varepsilon_z | \mu \in (V^{\Lambda_0})^*, z \in \Omega\}|_{\mathcal{H}_\gamma^{\pi_0}}$ spans $(\mathcal{H}_\gamma^{\pi_0})^*$. Thus $\phi_\gamma^{\pi_0,\lambda}$ is a linear combination of functions of the form $g \to (\mu \otimes \varepsilon_z)(E_\gamma T_{\pi_0,\lambda}(g) f)$. This proves the lemma.

LEMMA 3.7. *Let $\mathfrak{D}'(G)$ be the space of distributions on $C_0^\infty(G)$ with the weak topology. If $f \in C_0^\infty(G)$ define*

$$\theta_{\pi_0,\lambda}(f) = \sum_{\gamma \in K_0} \int_G \phi_\gamma^{\pi_0,\lambda}(g) f(g)\, dg.$$

Then the series defining $\theta_{\pi_0,\lambda}$ converges absolutely and uniformly on compact subsets of C. Furthermore the function $\lambda \to \theta_{\pi_0,\lambda}$ defines a holomorphic function from C to $\mathfrak{D}'(G)$.

PROOF. Let $\mathfrak{a} = \sum_{i=1}^r \mathbf{R}(X_{\gamma_i} + X_{-\gamma_i})$. Let $A = \exp \mathfrak{a}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$ corresponding to $(\mathfrak{k}, \mathfrak{a})$. Then $G = KAN$ is an Iwasawa decomposition of G. Let $\tilde{\rho}(H) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}\, H|_\mathfrak{n})$ for $H \in \mathfrak{a}$. Then, if f is integrable on G, we have

$$\int_G f(g)dg = \int_{K\times A\times N} f(kan) e^{2\tilde{\rho}(\log a)}\, dk\, da\, dn \tag{$*$}$$

where log: $A \to \mathfrak{a}$ is the inverse map to exp: $\mathfrak{a} \to A$.

Now $\mathfrak{k} = \mathbf{R}iH_1 \oplus \mathfrak{k}_1$, $\mathfrak{k}_1 = [k, k]$. Let K_1 be the connected subgroup of K corresponding to $\mathfrak{k}_1$. Then K_1 is compact and simply connected and the map $\mathbf{R} \times K_1 \to K$ given by $(t, k_1) \mapsto \exp(tiH_1)k_1$ is a Lie isomorphism of $\mathbf{R} \times K_1$ with K. We therefore see that

$$\int_G f(g)\, dg = \int_\mathbf{R}\left(\int_{K_1\times A\times N} f(\exp(t_i H_1)k_1 an) e^{2\tilde{\rho}(\log a)}\cdot dk_1 da\, dn\right) dt$$

for f absolutely integrable on G. Now let $f \in C_0^\infty(G)$. Then

$$\int_G \phi_\gamma^{\pi_0,\lambda}(g) f(g) dg = \int_\mathbf{R}\left(\int_{K_1\times A\times N} \phi_\gamma^{\pi_0,\lambda}((\exp t(iH_1))k_1 an)\right.$$
$$\left.\cdot f((\exp t(iH_1))k_1 an) e^{2\tilde{\rho}(\log a)}\cdot dk_1 da\, dn\right) dt.$$

Now

$$T_{\pi_0,\lambda}(\exp(tiH_1)k_1 an)|_{\mathcal{H}_\gamma^0} = e^{it(\lambda\Lambda_1(H_1)+\lambda_\gamma)} T_{\pi_0,\lambda}(k_1 an),$$

where $\pi_\gamma(\exp tiH_1) = e^{i\lambda_\gamma t}$ for $t \in \mathbf{R}$ (see Lemma 2.5). Hence we have

$$\int_G \phi_\gamma^{\pi_0,\lambda}(g) f(g)\, dg = \int_\mathbf{R} e^{it(\lambda\Lambda_1(H_1)+\lambda_\gamma)} \int_{K_1\times A\times N} \phi_\gamma^{\pi_0,\lambda}(k_1 an)$$
$$\cdot f(\exp(t(iH_1))k_1 an) e^{2\tilde{\rho}(\log a)}\, dk_1\, da\, dn.$$

If $\mu \in C$ and $\delta \in \hat{K}_1$ define

$$\tilde{f}(\mu : \delta : k_1 an) = d(\delta)\int_\mathbf{R} e^{i\mu t}\left(\int_{K_1} \overline{\chi_\delta(k)}\, f((\exp(itH_1))k^{-1}k_1 an) dk\right) dt.$$

Let $\|\delta\|$ be the norm of the highest weight of δ. Now $f(\mu : \delta : an)$ has compact support on AN. If $\omega \subset C$ is a compact subset of C and $l_1, l_2 > 0$ then

$$|\phi_\gamma^{\pi_0,\lambda}(k_1 an)\tilde{f}(\lambda\Lambda_1(H_1) + \lambda_\gamma : \gamma_0 : k_1 an)| < C_{l_1,l_2,\omega}(f)(1 + |\lambda + \lambda_\gamma|^2)^{-l_1}(1 + |\gamma_0|^2)^{-l_2}\psi_f(an)$$

with $\psi_f \in C_0^\infty(AN)$ and $f \to C_{l_1,l_2,\omega}(f)\psi_f(an)$ a continuous function from $C_0^\infty(G) \to C$. Here γ_0 is the restriction of γ to K_1. These estimates follow the Paley-Wiener theorem for $\mathbf{R}$ and K_1. This implies that

$$\sum_{\gamma \in K_0} \left| \int_G \phi_\gamma^{\pi_0,\lambda}(g) f(g)\, dg \right| < C'_{l_1,l_2,\omega}(f) \sum_{\gamma \in K} (1 + |\lambda\Lambda_1(H_1) + \lambda_\gamma|^2)^{-l_1}(1 + |\gamma_0|^2)^{-l_2}$$

for $\lambda \in \omega$ with $f \to C'_{l_1,l_2,\omega}(f)$ continuous on $C_0^\infty(G)$. This clearly implies the lemma (cf. Wallach [10, Chapter 5]).

COROLLARY 3.8. *The character of a "limit of holomorphic discrete series" is the limit of the characters of holomorphic discrete series. That is*

$$\lim_{\substack{\lambda \to C(\pi_0) \\ \lambda < C(\pi_0)}} \theta_{\pi_0,\lambda} = \theta_{\pi_0,C(\Lambda_0)},$$

and, since the representations $(T_{\pi_0,\lambda}, H^{\pi_0,\lambda})$, $\lambda < C(\Lambda_0)$, *and* $(T_{\pi_0,C(\Lambda_0)}, H^{\pi_0})$ *are trace class this implies the statement.*

Note. This implies that the Harish-Chandra formula [4] for the character of holomorphic discrete series is also true for limits of discrete series. Lemma 3.7 also says that the "signs" involved in the formula can be chosen so that the formula is holomorphic in λ where $\Lambda = \Lambda_0 + \lambda\Lambda_1$. This also allows one to compute characters of unitary representations which are "past the limit of holomorphic discrete series" (see Lemma 3.5).

4. Example: The universal covering group of $SU(n, 1)$. In this case we may identify $\mathfrak{p}^+$ with C^n and Ω with the unit ball in C^n: $\Omega = \{z \in C^n | \Sigma |z_i|^2 < 1\}$. If $g \in SU(n, 1) = G_0$ then

$$g = \begin{bmatrix} A & b \\ c^* & d \end{bmatrix}$$

with A, $n \times n$; c, b, $n \times 1$; $d \in C$. The condition that $g \in G_0$ is

$$gJ{}^t\bar{g} = J, \qquad \det g = 1$$

with

$$J = \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix},$$

I, $n \times n$. The action of G_0 on Ω (which we write on the *left*) is

$$g \cdot z = (\langle z, c\rangle + d)^{-1}(Az + b).$$

Actually G_0 acts on $\overline{\Omega}$. K_0 is the subgroup of G_0 consisting of the matrices

$$\begin{bmatrix} u & 0 \\ 0 & (\det u)^{-1} \end{bmatrix},$$

$u \in U(n)$. In particular $U(n)$ acts on $\mathfrak{p}^+ = C^n$ by

$$u \cdot z = (\det u) uz$$

where Uz is the usual action of $U(n)$ on C^n. It is classical that the representation of $U(n)$ on $\mathcal{P}^j(C^n)$ given by

$$(u \cdot f)(z) = f(u^{-1} \cdot z)$$

is irreducible.

We consider the case $\pi_0 = 1$. For $f \in \mathcal{P}^j(C^n)$

$$\int_\Omega e^{\lambda\Lambda_1}(D(z : z)) f(z) \overline{f(z)}\, d\mu(z) = c_j(\lambda)\langle f, f\rangle$$

since $\mathcal{P}^j(C^n)$ is irreducible. It is therefore convenient to take $X_{\gamma_1} = e_1 \in \mathbf{C}^n$ ($e_1, \ldots, e_n$ the standard basis of $\mathbf{C}^n$) and $f(z) = Z_1^j$. From this we see that

$$\int_\Omega e^{\lambda\Lambda_1}(D(z : z)) f(z) \overline{f(z)}\, d\mu(z)$$

$$= \int_0^1 t^{2n-1} \int_{U(n)} e^{\lambda\Lambda_1}\big(D(tX_{\gamma_1} : tX_{\gamma_1})\big)\, |f(\mathrm{Ad}(k)tX_{\gamma_1})|^2\, dk\, \frac{dt}{(1 - t^2)^{n+1}}$$

$$= \int_0^1 t^{2n-1}(1 - t^2)^{-\lambda\Lambda_1(H_{\gamma_1}) - n - 1} \left(\int_{U(n)} |B(\mathrm{Ad}(k)tX_{\gamma_1}, X_{-\gamma_1})|^{2j}\, dk \right) dt$$

$$= \frac{1}{d_j} \int_0^1 t^{2n-1}(1 - t^2)^{-\lambda\Lambda_1(H_{\gamma_1}) - n - 1} t^{2j}\, dt.$$

(Here we have used the orthogonality relations for $U(n)$ and $d_j = \dim S^j(\mathfrak{p}^+)$.)

$$= \frac{1}{2d_j} B\big(n + j, -\lambda\Lambda_1(H_{\gamma_1}) - n\big)$$

where $B(z, \omega)$ is the classical beta function (cf. Whittaker and Watson [11]). Now $\Lambda_1(H_{\gamma_1}) = 1$. We therefore have the following lemma.

LEMMA 4.1. *If* $G_0 = SU(n, 1)$ *and* $\pi_0 = 1$ *then* $C(1) = -n$. *That is* $T_{1,\lambda}$ *is holomorphic discrete series for* $\lambda < -n$. *Furthermore*

$$A_j(\lambda) = \frac{1}{2d_j} B(n+j, -\lambda - n)I = \left(\frac{1}{2d_j}\right) \frac{(n+j-1)!}{\prod_{s=1}^{n+j}(-\lambda+j-s)} I$$

$$= \frac{(-1)^{n+j-1}}{2d_j}\left(\frac{(n+j-1)!}{\prod_{s=0}^{n+j-1}(\lambda+n-s)}\right)I.$$

In particular

$$d(1,\lambda) = \frac{(-1)^{n-1}}{2\prod_{s=0}^{n-1}(\lambda+n-s)}$$

(see Lemma 3.2).

COROLLARY 4.1.

$$d(1,\lambda)^{-1}A_j(\lambda) = \frac{(-1)^j(n+j-1)!}{d_j}\left(\prod_{s=0}^{j-1}(\lambda - s)\right)^{-1} \cdot I.$$

COROLLARY 4.2. *If $f_1, f_2 \in \mathcal{P}(C^n)$ then $d(1,\lambda)^{-1}\langle f_1, f_2\rangle_{1,\lambda}$ defines a positive definite $T_{1,\lambda}$ invariant inner product on $\mathcal{P}(C^n) = \mathcal{H}_F^1$ for $\lambda \in \mathbf{R}$, $\lambda < 0$. These representations extend to unitary representations of G on the Hilbert space completion $H^{1,\lambda}$ of $\mathcal{P}(C^n)$.*

Note. The representations $-n < \lambda < 0$ are the direct generalization of the "extra representations" of Sally [8] that go "past" the limit holomorphic relative discrete series of the universal covering group of $SL(2, \mathbf{R})$.

We also note that $(T_{1,-n}, H^{1,-n})$ is the Hardy space for Ω.

LEMMA 4.2. *There is a constant $C > 0$ so that if $f \in \mathcal{P}(C^n)$ then*

$$\lim_{\substack{\lambda\to -n \\ \lambda < -n}} d(1,\lambda)^{-1}\|f\|_{1,\lambda}^2 = \lim_{t\to 1} C\int_{S^{2n-1}} |f(t\omega)|^2\, d\omega,$$

where $d\omega$ is invariant measure on S^{2n-1}.

PROOF. This result follows directly from Corollary 4.2, Corollary 2.7 and the results of Knapp-Okamoto [6].

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